Dynamic Viscosity Measurement Method Based on the Stokes Drag of Prolate Ellipsoidal Mass

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Abstract. *Viscometer plays an important role in the field of tribology. One way to measure viscosity is to use the Stokes drag principle in the underdamped harmonic oscillation phenomenon. This paper proposes a dynamic viscosity measurement method based on the related physical laws. The experimental model involves a prolate ellipsoidal mass that experiences underdamped harmonic oscillation within viscous liquid samples. The oscillations of the prolate ellipsoid were observed to obtain the viscous damping coefficient of each sample, which was then substituted to the theoretical formula of dynamic viscosity. Experimental data suggest that the mathematical model has failed to predict the viscosity values of the samples. In addition, the regression curve of the reference viscosity and the measured viscous damping coefficient shows that the two quantities have an exponential relation instead of linear relation as explained in the theoretical model. In this work, the regression formula is considered as the empirical measurement transfer function, which is used to measure the viscosity of an ISO VG 150 industrial oil sample. This measurement resulted in a 2.40 % of relative error percentage. Lastly, this measurement method is only valid for measuring samples with viscosities ranging from 0.040 Pa s to 0.256 Pa s.*

Keywords: *viscosity, Stokes drag, damping coefficient, oscillation, prolate ellipsoid.*

Abstrak. Viskometer memainkan peranan penting pada bidang tribologi. Salah satu cara untuk mengukur viskositas adalah dengan menggunakan prinsip hambatan Stokes pada fenomena osilasi harmonik teredam. Artikel ini mengajukan sebuah metode pengukuran viskositas dinamik cairan berdasarkan hukum-hukum fisika terkait. Model eksperimen pada penelitian ini melibatkan sebuah massa berbentuk elipsoid prolat yang mengalami osilasi harmonik teredam di dalam sampel cairan. Gerak osilasi elipsoid prolat diamati untuk memperoleh koefisien redaman tiap sampel cairan, yang kemudian disubtitusikan pada persamaan teoritik viskositas dinamik. Data eksperimental yang diperoleh menunjukkan bahwa model matematik yang digunakan telah gagal dalam memperediksi nilai viskositas tiap sampel. Sebagai bukti, kurva regresi viskositas referensi terhadap data koefisien redaman menunjukkan bahwa kedua-nya memiliki hubungan eksponensial dan bukan hubungan linear seperti yang dijelaskan pada model teoritik. Persamaan regresi yang diperoleh dianggap sebagai fungsi transfer empirik, yang digunakan untuk mengukur viskositas sampel oli industri ISO VG 150. Pengukuran ini menghasilkan persentase kesalahan relatif sebesar 2,40 %. Sebagai penutup, metode pengukuran ini hanya valid untuk pengukuran sampel yang memiliki nilai viskositas pada jangkauan 0,040 Pa s hingga 0,256 Pa s.

Kata kunci: viskositas, hambatan Stokes, koefisien redaman, osilasi, elipsoid prolat.

INTRODUCTION

The viscously damped spring-mass oscillator system is a simple but helpful mechanical model which is commonly used to study the phenomena of mechanical vibrations [1]. Theoretically, the damping coefficient of an underdamped harmonic oscillation within a viscous liquid is directly proportional to the liquid viscosity and also determined by the shape constant of the mass that performs the oscillatory motion [2]. The viscous damping coefficient value of a certain liquid can be experimentally obtained using an exponential regression of the mass displacement during the underdamped oscillation [3]. Since the Stokes law explains the linear relation between retarding force experienced by a settling sphere under viscous liquid and the dynamic viscosity of the liquid, it is possible to obtain a simple mathematical model involving dynamic viscosity and viscous damping coefficient [4].

A few researches which studied the use of Stokes drag and damped harmonic oscillation principles to measure fluid viscosity had been done before. Peter and Evi (2004) measured the viscosities of three liquid samples using the linear relation between viscous damping coefficient and dynamic viscosity [5]. Their design of oscillator system consists of a spring and a spherical mass placed within the liquid container. Sohaib, Wasif and Muhammad (2011) did a similar study, but they added some important parameters into the mathematical model such as oscillation angular frequency, liquid density, and the radius of the cylindrical liquid container [6]. Finally, Oki and Yudistira (2017) attempted air viscosity measurements using the same principles, but with different arrangements [7]. They used a cylindrical graphite rod which levitates above a pair of dipole magnets as the oscillating mass and extracted the damping time constant to measure air viscosity.

In this paper, a similar method used to measure liquid dynamic viscosity is presented. The novelty of this research lies on the shape of the mass which encounters viscous drag during the underdamped oscillatory motion, because a prolate ellipsoidal mass is chosen to do the work. Thus, an appropriate mathematical model of dynamic viscosity is presented in this paper. Since viscosity is a major physical parameter in engine lubricants and industrial oil, this measurement method might be useful in the tribological industry.

Theory

A sinking mass that moves with a constant velocity v through a liquid column encounters a resisting force F_d that can be mathematically expressed as:

$$F_d = -6\pi \mu r K v \qquad (1)$$

where μ is the dynamic viscosity of the liquid, r is the radius of the object which is perpendicular to the motion axis, while K is the shape factor of the mass. K is a unity if the mass is a sphere, but for a prolate ellipsoid which moves through a viscous fluid column along its semi-major axis, the appropriate shape factor obeys the equations below [8]:

$$K = \frac{4}{3\sqrt{\alpha^2 - 1}\ [(\alpha^2 + 1)(\coth^{-1}\alpha) - \alpha]} \qquad (2)$$

$$\alpha = \frac{1}{\sqrt{1 - \left(\frac{r_2}{r_1}\right)^2}} \qquad (3)$$

where α is some constant related with the prolate ellipsoid's dimension, while r_1 and r_2 are respectively the semi-major and the semi minor axes of the prolate ellipsoid as shown in **Figure 1 (a)**.

By considering the prolate ellipsoid as an oscillator mass which hangs on the bottom of a spring within viscous liquid as in **Figure 1 (b)**, the drag experienced by the prolate ellipsoid during the underdamped oscillation also obeys [9]:

$$F_d = -bv \qquad (4)$$

where b is the viscous damping coefficient of the oscillatory motion.

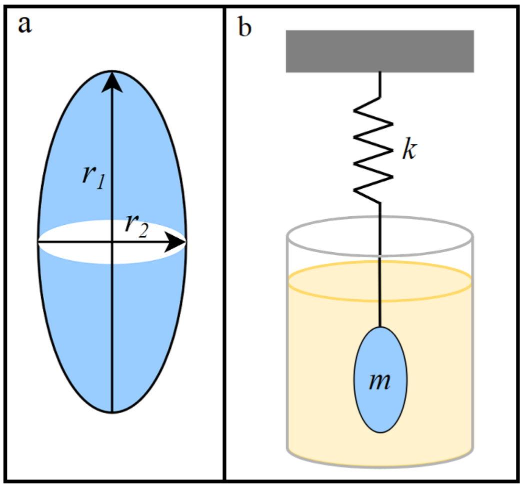

Figure 1. (a) Prolate ellipsoid schematic diagram. **(b)** Spring-mass oscillator system Schematic diagram.

The minus sign in the right side of **Equation 4** suggests that the retarding force opposes the direction of the motion. Finally by substituting **Equation 4** to **Equation 1**, the formula needed for viscosity measurements is obtained:

$$b = 6\pi\mu r K \tag{5}$$

As the mass is being displaced by an amount y at time t during the underdamped oscillation, the system's general equation of motion can be written as:

$$m\frac{dy^2}{dt^2} = -b\frac{dy}{dt} - ky \tag{6}$$

where m is the mass of the prolate ellipsoid and k is the spring constant. The appropriate solution for **Equation 6** would be the displacement equation as a function of time:

$$y(t) = Y_0 e^{\left(-\frac{bt}{2m}\right)} \cos \omega t \tag{7}$$

where Y_0 is the displacement amplitude and ω is the oscillation angular frequency. The term $Y_0 \exp{(-bt/2m)}$ is the exponential amplitude decay of the oscillatory motion and can be used as the exponential regression function of the experimental data.

RESEARCH METHODS

Experimental Setup

An incremental optical encoder is chosen as the oscillatory motion detector due to its ability to encode the rotation of the shaft into a pair square wave signals with 90^0 phase difference [10]. The data acquisition system collects the output pulses of the encoder during the oscillation. Then, the time series of mass displacements are extracted based on the square wave signals.

A string passes through the side of the encoder shaft so that the shaft will be able to rotate as the mass performs underdamped harmonic oscillation within the viscous liquid. The oscillator schematic diagram is shown in **Figure 2**, where there is a ball bearing that is used to bend the string and point it onto the encoder shaft. For conveniences, it is assumed that the string is massless and the bearing mass has no effect to the oscillatory motion.

Clearly, the stokes law of viscous drag only applies in liquid flows with very low Reynolds number [11]. In order to prevent turbulent flows during the experiment, a prolate ellipsoidal lead with a smooth surface is chosen as the oscillating mass.

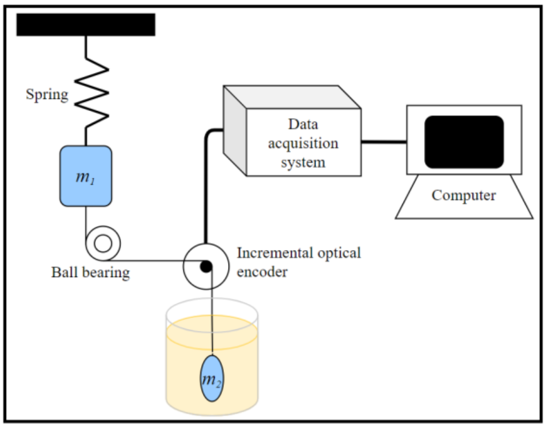

Figure 2. System schematic diagram.

In order to prevent turbulent flow during the oscillation, the size of m_2 is minimized, with the semi-major axis $r_1 = 3.100$ cm and semi-minor axis $r_2 = 0.815$ cm. However, this leaves the prolate ellipsoid with a mass of $m_2 = 95.0$ g.

The mass of the ellipsoid is too small to generate a significant amount of spring displacement, since the spring constant k was experimentally obtained with a value of 82.8 N/m. It was then decided to add an extremely larger mas m_1 which is cylindrical in shape into the system, where $m_1 = 1.50$ kg. Another purpose of m_1 is to overcome the net resisting force generated by the friction of the ball bearing and the encoder shaft rotary system as well as air resistance during the underdamped harmonic oscillation.

Liquid Samples

It was decided to measure the viscosities of six industrial oil samples, with the viscosity grades: ISO VG 32, ISO VG 46, ISO VG 68, ISO VG 100, ISO VG 220 and ISO VG 320. Those grades are set by the International Organization for Standardization (ISO), where the larger the Viscosity Grade (VG), the more viscous the liquid is. The six samples were tested under a constant temperature of 40 °C, because the reference viscosities of those liquids are displayed at that temperature value as reflected in **Table 1** [12].

Table 1. Sample reference kinematic viscosity values.

Liquid sample viscosity grade	Reference kinematic viscosity (at 40 °C)
ISO VG 32	32 ± 3.2 mm²/s
ISO VG 46	46 ± 4.6 mm²/s
ISO VG 68	68 ± 6.8 mm²/s
ISO VG 100	100 ± 10 mm²/s
ISO VG 220	220 ± 22 mm²/s
ISO VG 320	320 ± 3.2 mm²/s

The proposed method can only be used to measure liquid dynamic viscosity, while the sample specification in **Table 1** displays reference kinematic viscosity values. In order to be able to measure the sample viscosity in the appropriate way, the reference kinematic viscosity η_{ref} values were converted into dynamic viscosities μ_{ref} using a simple equation [13]:

$$\mu_{ref} = \eta_{ref}\ \rho_f \quad (8)$$

where ρ_f is the density of the liquid.

Experimental Procedure

First, the damping coefficient that is generated purely by the system's mechanical friction and the air resistance is measured by recording the underdamped harmonic oscillation with no liquid sample. Then the data is fitted with the exponential amplitude decay regression curve:

$$y_{fit} = Y_0 e^{\left(-\frac{bt}{2m}\right)} \quad (9)$$

where m is the total mass of m_1 and m_2. The b value that best represents the data is picked as the experimental damping coefficient b_0 of the mechanical system. Then, the damping coefficient b_1 was measured by recording the underdamped harmonic oscillation within each liquid sample through the exact same steps.

It is assumed that the total retarding force experienced by the prolate ellipsoidal mass while oscillating is a net force of viscous drag, mechanical friction of the oscillator components and also air resistance. For conveniences, it is considered that the resultant of those resisting forces (excluding the viscous drag due to the liquid) can be interpreted as another viscous drag, and therefore it also obeys **Equation 4**. Thus, b_1 was subtracted with b_0 to get b_2, which is the damping coefficient value that is purely generated by the viscous drag during the oscillation. In order to obtain the viscosity measurement results, the b_2 value of each sample was then substituted to the viscosity mathematical model obtained from the derivation of **Equation 2** and **Equation 5**:

$$\mu = \frac{b\,\sqrt{\alpha^2-1}\;[(\alpha^2+1)\,(coth^{-1}\,\alpha)-\alpha]}{8\pi r_2} \qquad (10)$$

Finally, the experimental dynamic viscosity μ of each sample is compared to the reference dynamic viscosity μ_{ref} to obtain the accuracy of this measurement method.

RESULTS AND DISCUSSION

In this part, the results of the viscous oscillation tests which were applied to the six samples is presented. First of all, ten repetitions of oscillator damping coefficient b_0 measurement were done. An average value of $b_0 = 0.495$ Ns/m was obtained with a relative standard deviation of 2.10%. An ideal oscillator system has a b_0 value of 0, which means that the system is free from any factors which damp the oscillation except for the viscous drag. This experimental fact suggests that the total retarding force caused by the friction of the system's mechanical components combined with air resistance generates significant contribution to the oscillation damping effect.

Next, ten repetitions of the oscillation test for each of the six liquid samples were done exactly like the liquid-less oscillation test. For every sample, the average value $\bar{b}_2$ is taken as the experimental viscous damping coefficient of the corresponding liquid. **Figure 3** shows one of the ten oscillation curves of each industrial oil sample. The experimental average viscous damping coefficient $\bar{b}_2$ and the relative standard deviation value of each sample measurement repetitions are shown in **Table 2**, which suggests that viscosity is proportional to damping coefficient.

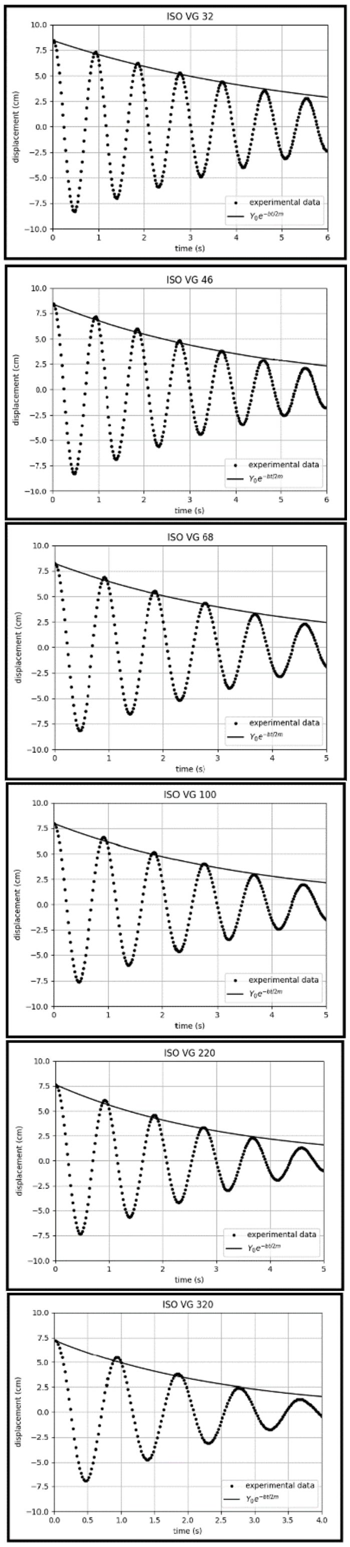

Figure 3. Underdamped oscillation test results.

Table 2. Experimental average damping coefficient and relative standard deviation values of each sample.

Liquid sample	$\bar{b}_2$ (N s m^{-1})	Relative standard deviation
ISO VG 32	0.062	18.60 %
ISO VG 46	0.200	5.540 %
ISO VG 68	0.278	3.840 %
ISO VG 100	0.328	3.180 %
ISO VG 220	0.508	1.900 %
ISO VG 320	0.726	1.700 %

Table 3. Viscosity measurement results.

Liquid sample (ISO VG)	μ_{ref} (Pa s)	μ_{meas} (Pa s)
32	0.027 ± 0.003	0.258 ± 0.048
46	0.040 ± 0.004	0.835 ± 0.046
68	0.061 ± 0.006	1.160 ± 0.045
100	0.087 ± 0.009	1.370 ± 0.044
220	0.191 ± 0.019	2.120 ± 0.040
320	0.256 ± 0.026	3.030 ± 0.052

Applying those $\bar{b}_2$ values to **Equation 10** results in the measured dynamic viscosity μ_{meas} value of each sample. **Table 3** displays the viscosity measurement results.

The measured liquid viscosities have extremely larger values than the reference viscosities, which means that the theoretical model has failed to predict the dynamic viscosities of the six samples. **Figure 4** Displays the regression model of the measured viscosity as a function of reference viscosity. The linear regression below is far from the ideal model, since the measured viscosity values are expected to be equal to the reference viscosities.

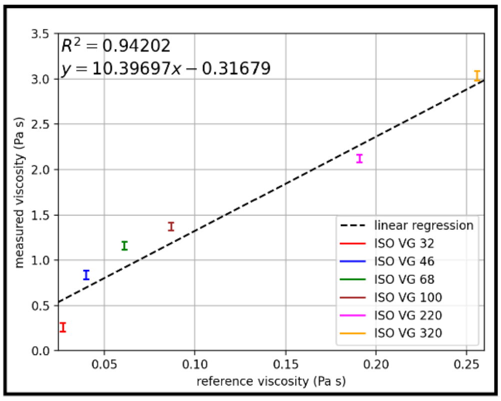

Figure 4. Linear regression of measured viscosity based on the theoretical formula as a function of reference viscosity.

There are two main aspects that cause these measurement errors. First, the viscous drag model in **Equation 1** that was derived only applies for a system where the oscillating mass moves with a constant velocity within a viscous fluid without experiencing any oscillatory motion [14]. The system used in this research denies these conditions, because the prolate ellipsoidal mass oscillates with an underdamped manner and its velocity changes during the oscillatory motion.

Second, the ideal environment of a viscous underdamped harmonic oscillation is an unbounded fluid, since finite fluid boundary generates an increase on the retarding force which may reduce the settling velocity of the mass [15]. In this work, a metal cylindrical pipe with one opened end is used as the viscous liquid container which is 2.42 cm in radius and 30.0 cm in height. The finite size of the cylindrical container allows wall retardation which results on inaccurate damping coefficient measurements [16].

The empirical transfer function which explains the relationship between reference dynamic viscosity and viscous damping coefficient was then decided to be obtained from the experimental data. That formula will then be used to measure the viscosity of another liquid sample.

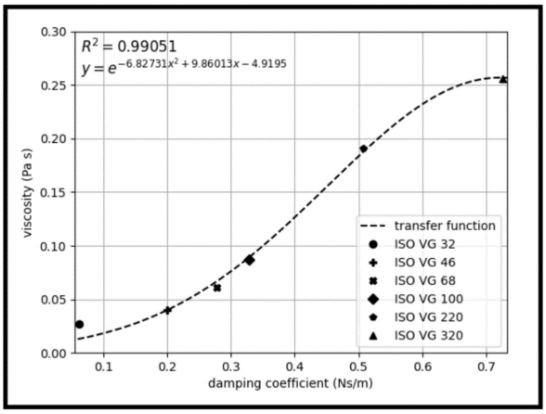

Figure 5. Measurement transfer function curve.

The transfer function which is a regression of the reference viscosity as a function of viscous damping coefficient is shown in **Figure 5**. The most appropriate transfer function model that was acquired from the regression is a second order polynomial exponential formula:

$$\mu_{meas} = e^{(-6.827b^2+9.860b-4.920)} \qquad (11)$$

To test the reliability of the empirical formula above, the dynamic viscosities of the six samples were calculated. **Table 4** displays the new measurement results obtained using **Equation 11**. The new measurement results show a significant difference from ones obtained using the theoretical formula as shown in **Table 3**.

Table 4. Dynamic viscosity measurement results based on the system transfer function.

Liquid Sample	μ_{ref} (Pa s)	μ_{meas} (Pa s)
ISO VG 32	0.027 ± 0.003	0.013 ± 0.001
ISO VG 46	0.040 ± 0.004	0.040 ± 0.003
ISO VG 68	0.061 ± 0.006	0.064 ± 0.004
ISO VG 100	0.087 ± 0.009	0.087 ± 0.005
ISO VG 220	0.191 ± 0.019	0.190 ± 0.006
ISO VG 320	0.256 ± 0.026	0.256 ± 0.001

The measured viscosity μ_{meas} values which were calculated using the transfer function are much closer to the reference viscosity μ_{ref} values.

Although they seemed to be good overall measurement results, it was spotted that the ISO VG 32 μ_{meas} value has considerable deviation from the μ_{ref} value. Clearly the Stokes drag generated by the ISO VG 32 sample viscosity is too weak, so that most of the damping effect appeared during the harmonic oscillation comes from external factors which are the system's mechanical friction and the air resistance. In other words, this measurement system does not work well on liquids which generate relatively small $\bar{b}_2$ values such as the ISO VG 32 oil. The ISO VG 32 data was then removed from the regression curve in order to get a better measurement transfer function. **Figure 6** shows the graphical visualization of the new measurement transfer function.

The second measurement transfer function was obtained from the same form of regression, but it has a higher coefficient of determination R^2 value than the original transfer function. Thus, this transfer function is expected to be more accurate than the previous one. The new empirical equation of dynamic viscosity can be written as:

$$\mu_{meas} = e^{(-7.529b^2+10.662b-5.134)} \qquad (12)$$

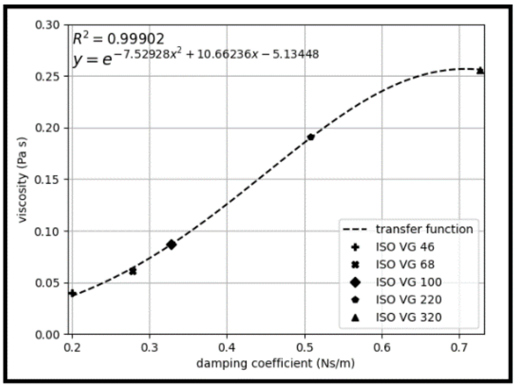

Figure 6. The second measurement transfer function curve.

Table 5. Dynamic viscosity measurement results based on the new system transfer function.

Liquid Sample	μ_{ref} (Pa s)	μ_{meas} (Pa s)
ISO VG 46	0.040 ± 0.004	0.037 ± 0.003
ISO VG 68	0.061 ± 0.006	0.064 ± 0.004
ISO VG 100	0.087 ± 0.009	0.087 ± 0.005
ISO VG 220	0.191 ± 0.019	0.190 ± 0.006
ISO VG 320	0.256 ± 0.026	0.256 ± 0.001

Equation 12 is used to calculate the dynamic viscosities of ISO VG 46, ISO VG 68, ISO VG 100, ISO VG 220 and ISO VG 320 samples. **Table 5** displays the measurement results which were obtained using the second transfer function, while **Figure 7** visualizes the regression model of measured viscosity as a function of reference viscosity.

Ideal viscosity measurements will generate results which have the same values as the reference viscosities. **Figure 7** shows the obtained gradient value of 1.001 which is extremely close to 1, and a regression constant ($y_{intercept}$) of 0.001 that is approximate to 0. To conclude, the measurement results are accurate for liquids with the viscosities within the range of 0.040 to 0.256 Pa s.

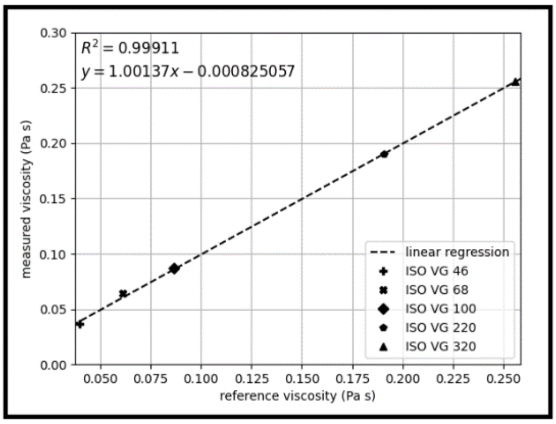

Figure 7. Linear regression of measured viscosity based on the new measurement transfer function.

It was agreed to consider the new transfer function in **Equation 12** as the accepted dynamic viscosity measurement formula of the system. In order to test its reliability, it was used to measure the viscosity of another sample.

The ISO VG 150 industrial oil is picked as the new sample, which has a reference viscosity of 0.135 ± 0.014 Pa s (at 40 °C). Ten more repetitions of oscillation test for the ISO VG 150 sample at the same temperature point were done. The average viscous damping coefficient $\bar{b}_2$ value that was obtained is 0.410 N s m^{-1}. Substituting it to **Equation 12**, it leaves a viscosity measurement value μ_{meas} of 0.132 ± 0.007 Pa s. This measured value is in good agreement with the reference viscosity and yields a relative error percentage of 2.40 %.

CONCLUSION

A viscosity measurement method is proposed in this paper. The underdamped harmonic oscillatory motion of a prolate ellipsoid in viscous liquids was observed and the liquid viscous damping coefficients were obtained. The proposed theoretical formula of dynamic viscosity is derived from the general form of Stokes drag equation. Six industrial oil samples were tested under the same treatment for viscosity measurement purposes. Since the ideal conditions of viscous damping phenomenon were not met in the experimental setup, the theoretical model has failed to predict the viscosity values of the six samples. The measurement transfer function is considered as the empirical formula of dynamic viscosity, after eliminating the influence of ISO VG 32 experimental data from the regression curve. The accuracy of the accepted transfer function was tested by measuring an ISO VG 150 oil sample, which leaves a good result. The ISO VG 150 oil has a reference viscosity of 0.135 ± 0.014 Pa s at 40 °C, and a measured viscosity value of $0.132 \pm$

0.007 Pa s is obtained at the exact same temperature point. This measurement results in a considerably small relative error percentage of $2.40\ \%$. Lastly, since the transfer function is obtained from the experimental damping coefficients of ISO VG 46 to ISO VG 320, this method can only be used to measure the viscosities of liquids within inside the range of 0.040 Pa s to 0.256 Pa s.

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